

The Density Temperature and the Dry and Wet Virtual Adiabats

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ABSTRACT

A density temperature is introduced to represent virtual temperature and potential temperature on thermodynamic diagrams. This study reviews how the dry and wet virtual adiabats can be used to represent stability and air parcel density for unsaturated and cloudy air, and present formulae and tabulations.

1. Introduction

Betts (1982, 1983) introduced the concept of dry and wet virtual adiabats on thermodynamic diagrams to include graphically the density effects of water substance. He suggested they were more useful reference processes than the dry and wet adiabats for the study of stability in the atmosphere. The purpose of this paper is to extend the discussion by introducing a *density temperature*, and to present formulae and tables for these reference processes to assist data analysis and forecasting.

Betts and Albrecht (1987) noted that below cloud base the mean thermodynamic structure of convective boundary layer soundings over the tropical oceans follows a dry virtual adiabat. Betts (1986) suggested, based on observational studies, that the convective equilibrium lapse rate of the tropical atmosphere is close to a wet virtual adiabat from the surface to the freezing level. He proposed that the use of the wet virtual adiabat in a convective adjustment scheme would be superior to the wet adiabatic adjustment scheme suggested by Manabe et al. (1965). Subsequently, an observational study by Xu and Emanuel (1989), numerical studies by Cohen (1989), and Cohen and Frank (1989) have confirmed that the lower-tropical troposphere is close to neutrality with respect to the wet virtual adiabat.

2. Virtual temperature

The presence of water vapor and liquid water in an air parcel changes density and hence its buoyancy. The definition of the virtual temperature, T_v , is well known

(Saunders 1957; Deardorff 1980). It is the temperature of a *dry* parcel of air that has the same density as a given moist or cloudy air parcel. A good derivation of T_v can be found in Stull (1988, p. 645). For cloudy air, the virtual temperature, T_v , is given by

$$T_v = T(1 + 0.61q_s - l) \quad (1a)$$

where q_s is saturation mixing ratio and l is the mixing ratio of cloud water: both in kg kg^{-1} . For unsaturated air this reduces to

$$T_v = T(1 + 0.61q) \quad (1b)$$

where q is the water vapor mixing ratio.

For moist unsaturated air, T_v depends only on T and q ; a moist parcel is less dense (warmer in T_v) because of the water vapor content with a lower-molecular weight. The presence of liquid water in cloudy air makes a parcel more dense because the liquid has mass but negligible volume.

3. Unsaturated air

a. Virtual potential temperature

It is convenient to define a virtual potential temperature,

$$\theta_v = T_v(1000/p)^{R/C_p}, \quad (2)$$

so that $T_v/\theta_v = T/\theta$. The gas constant R and specific heat C_p can be taken as constants. As unsaturated air ascends adiabatically, θ_v is conserved, since $\theta_v/\theta = (1 + 0.61q)$ is also conserved. If air is lifted to the level where it is saturated, a set of conserved *saturation point* (SP) variables (Betts 1982) can be defined. An SP variable shall be denoted by a superscript *. As an unsaturated parcel is lifted adiabatically, T and p change,

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but θ and q do not, so that $\theta = \theta^*$, $q = q^*$, and $\theta_v = \theta_v^*$. A conserved T_v^* may be defined from

$$\begin{aligned} \theta_v &= \theta_v^* = T_v^* (1000/p^*)^{R/C_p} \\ &= T^* (1 + 0.61q^*) (1000/p^*)^{R/C_p}. \end{aligned} \quad (3)$$

Both the saturation level virtual temperature, (T_v^*), and the virtual potential temperature (θ_v^*) are conserved in adiabatic motion, since they are unique functions of (T^* , p^*) at the SP. Clearly, isolines of constant T_v^* and θ_v^* can be drawn on a thermodynamic diagram. Figure 1 shows these on a tephigram. A line of constant q^* is shown dotted. The lines of constant T_v^* (short dashes) are only slightly rotated from the isotherms because q^* only increases slowly with p at constant T . The lines of constant θ_v^* (heavy dashes) are rotated more from the dry adiabats (constant θ^*) (see section 3d), because q^* increases more rapidly with p at constant θ^* .

b. Density temperature

The density temperature $T_\rho(T, q)$ is now introduced. This is the temperature of an *exactly saturated* parcel that has the same *density* and virtual temperature at the same pressure level p as an *unsaturated* air parcel with temperature T , dewpoint T_D , and corresponding mixing ratio, q . Note that T_ρ is less than T because the air parcel is unsaturated and has $q < q_s$ (since it would be saturated at T_D).

$$T_v = T(1 + 0.61q) = T_\rho [1 + 0.61q_s(T_\rho)]. \quad (4)$$

In terms of potential temperatures, there are three relationships

$$\begin{aligned} \theta_v &= \theta(1 + 0.61q) = \theta^*(1 + 0.61q^*) = \theta_v^* \\ &= \theta_\rho(T_\rho, p) [1 + 0.61q_s(T_\rho, p)] \end{aligned} \quad (5)$$

where $\theta_\rho = T_\rho(1000/p)^{R/C_p}$.

Figure 1 shows T^* and T_ρ for $\theta_v^* = 300$ K, and their values at 800 and 900 mb are given in Table 1. For cloudy air, θ_v^* corresponds to a virtual evaporation or liquid water potential temperature, which represents the density of the parcel if its liquid water reevaporates.

The distinction between T_ρ and T_v is important. Both represent, at constant p , the density of an air parcel with water vapor (and in section 4, cloud water as well). However, T_v is the temperature of a *dry* parcel of the same density, while T_ρ is the temperature of a *just saturated* parcel of the same density. Since every point (T, p) on a thermodynamic diagram has an associated saturation mixing ratio, q_s , T_ρ is the more convenient representation of density on a thermodynamic diagram in terms of the temperature coordinates at a parcel pressure p . In particular, when an unsaturated parcel is lifted to saturation, T_ρ becomes just its saturation level temperature T^* . Although isolines of virtual temperature, T_v^* , can be added to a thermodynamic diagram as in Fig. 1, it is sufficient to plot the temperature, T_ρ , of just-saturated parcels of the same density. An analogy is useful; T_ρ is related to T_v , just as the dewpoint is related to the derived field of mixing ratio.

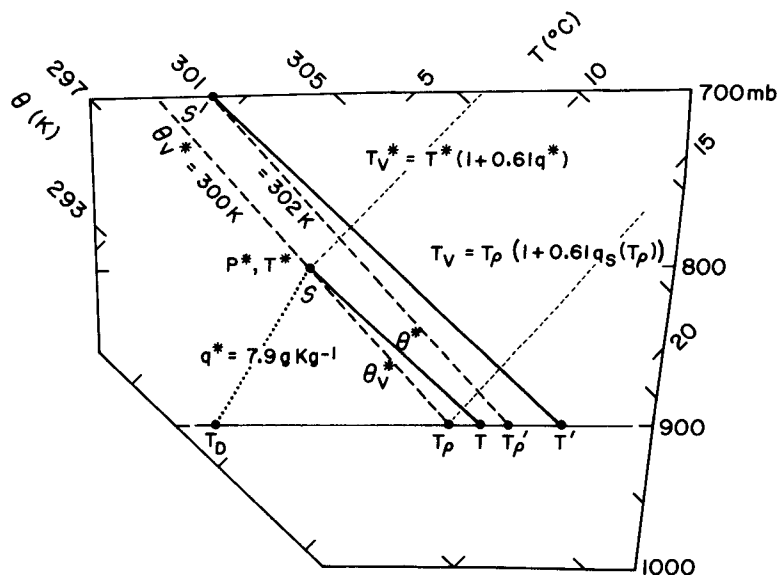


FIG. 1. Tephigram showing relationship of the density temperature (T_ρ), the dry virtual adiabat (the dashed isolines of constant θ_v^*) for two unsaturated air parcels with saturation points at S and S' , and the isolines of constant virtual temperature for saturated (but cloud-free) air parcels.

TABLE 1. Temperature (°C) on the dry virtual adiabats (constant θ_v^*).

p (mb)	θ_v (K)					
	260	270	280	290	300	310
100	-138.57	-133.39	-128.22	-123.04	-117.86	-112.18
200	-109.06	-102.75	-96.44	-90.13	-83.82	-77.51
300	-88.89	-81.80	-74.01	-67.63	-60.54	-53.46
400	-73.09	-65.39	-57.70	-50.01	-42.54	-34.68
500	-59.50	-51.71	-43.52	-35.35	-27.22	-19.15
600	-48.50	-39.87	-31.27	-22.72	-14.26	-5.95
700	-38.39	-29.40	-20.47	-11.63	-2.96	5.42
800	-29.28	-19.99	-10.79	-1.77	6.96	15.26
900	-20.99	-11.44	-2.06	7.05	15.73	23.83
1000	-13.36	-3.63	5.86	14.95	23.49	31.31

c. Dry virtual adiabats

Isolines of constant θ_v^* can be called *dry virtual adiabats*, by analogy with naming isolines of constant θ , the dry adiabats. In Fig. 1, as an unsaturated parcel is lifted adiabatically conserving θ_v^* , its density temperature and virtual temperature follow the dashed, dry virtual adiabat to the SP. Table 1 gives (density) temperature as a function of pressure on the dry virtual adiabats given by (5).

The construction shown in Fig. 1 can be used to plot the density temperatures (and the corresponding virtual temperatures) for a sounding by plotting T_p rather than T at each level. This is a similar convention to the one used for plotting the dewpoint. Figure 1 also shows how one can find and compare the density temperatures, T_p , T'_p of different unsaturated parcels brought adiabatically to the same pressure p . The dry virtual adiabats can be drawn through the parcel SPs S and S' in Fig. 1 [corresponding to the lifting condensation level (T^* , p^*) for each parcel] to the pressure level p , just as the corresponding dry adiabats can be drawn to find and compare the parcel temperatures T and T' . This construction can be regarded as an extension of Normand's Theorem for unsaturated parcels. That is, the dry adiabat through the parcel temperature, the wet adiabat through the wet-bulb temperature, the q isopleth through the dewpoint, and the dry virtual adiabat through the density temperature all converge to the saturation point.

If S' lies on the same dry virtual adiabat ($\theta_v^* = 300$ K) as S , then clearly both parcels would have the same density, when brought adiabatically to any pressure, where both are unsaturated. Thus, the dry virtual adiabat is the reference adiabat for comparing the *density* of unsaturated parcels with different SPs, and the *stability* of atmospheric layers with gradients of both θ and q . Unsaturated atmospheres, which are convectively mixed to near neutral stability, usually have an SP structure close to a dry virtual adiabat (Betts and

Albrecht 1987); that is, θ_v^* is nearly constant with height, although there is a gradient of θ and q .

d. Slope of the dry virtual adiabats

On a θ^* , q^* plot, the dry virtual adiabats are nearly linear. Differentiating (5) gives

$$(\partial\theta_v^*/\partial q^*)_{\theta^*} = -(1 + 0.61q^*)(\partial\theta^*/\partial q^*)_{\theta^*} \\ = 0.61\theta^* = 183 = 1/5.5 \times 10^{-3} \quad (6)$$

for $\theta^* = 300$ K. A convenient rule of thumb is that θ_v^* changes approximately 1 K every 5.5 g kg⁻¹ at constant θ^* or, alternatively, θ^* changes 1 K every 5.5 g kg⁻¹ along a dry virtual adiabat.

If the slope of the θ_v^* isopleth is derived in terms of (T , z), then it can be compared with the fixed slope of the dry adiabat. Rearranging (6) to

$$(1 + 0.61q^*)\delta\theta^* + 0.61\theta^*\delta q^* = 0 \quad (6')$$

and substituting for $\delta\theta^*$, δq^* in terms of δT and δp , and then using the hydrostatic and Clausius-Clapyron relationships, gives after rearrangement.

$$(\partial T_p^*/\partial z)_{\theta_v^*} = (-g/C_p)[1 + 0.61q^*(1 + C_p/R)]/ \\ [1 + 0.61(q^* + \alpha)] \\ = (-g/C_p)[1 + 0.61q^* + 0.61\alpha\epsilon/0.622)]/ \\ [1 + 0.61q^* + 0.61\alpha] \quad (7)$$

where

$$\alpha = T(\partial q_s/\partial T)_p \approx Lq_s/R_vT \quad \text{and} \quad \epsilon = C_pT/L. \quad (8)$$

The derivation is similar to finding the slope of the wet adiabat [(11) later]. Inserting values for g , C_p , R , R_v , and L gives for $T \approx 290$ K, with height in km,

TABLE 2. Dependence of slope of dry and wet virtual adiabats on q^* at 900 mb.

	q^* (g kg ⁻¹)						
	0.1	1	5	10	15	20	25
$(\partial T_p/\partial z)_{\theta_c^*}$ (K km ⁻¹)	-9.73	-9.64	-9.29	-8.92	-8.59	-8.30	-8.04
β_1 (dry)	.059	.075	.120	.166	.206	.242	.274
$(1 - \beta_2)$ (wet)	.907	.899	.896	.898	.901	.904	.907

$$(\partial T_p^*/\partial z)_{\theta_c^*} = -9.74(1 + 2.75q^*)/(1 + 12.0q^*). \quad (9)$$

For small moisture values, the dry virtual adiabats are asymptomatic to the dry adiabats (slope -9.74 K km⁻¹), but at the higher values of q^* found in the tropics their slope becomes appreciably less than that of the dry adiabat. Table 2 gives this slope at 900 mb, but the dependence on p is small.

e. Relation to the slope of the wet adiabats

A similar derivation to (7) gives the slope of the wet adiabat (a line of constant equivalent potential temperature θ_e^*) as

$$\left(\frac{\partial T}{\partial z}\right)_{\theta_e^*} = -\frac{g}{C_p}(1 + \alpha/0.622)/(1 + \alpha/\epsilon). \quad (10)$$

Reorganizing to give the slope in terms of θ gives

$$\begin{aligned} \Gamma_w &= (\partial\theta/\partial z)_{\theta_e^*} \\ &= (g\theta/C_p T)(\alpha/\epsilon - \alpha/0.622)/(1 + \alpha/\epsilon) \end{aligned} \quad (11)$$

Rewriting (7) in terms of θ and dividing by (11) gives after rearrangement

$$(\partial\theta_p/\partial z)_{\theta_e^*} = [0.61(\epsilon + \alpha)/(1 + 0.61q^* + 0.61\alpha)]\Gamma_w = \beta_1\Gamma_w. \quad (12)$$

Table 2 shows the coefficient β_1 . It is seen that at tropical mixing ratios, the dry virtual adiabat has a slope which is an appreciable fraction (>0.2) of that of the wet adiabat, Γ_w (Betts 1983).

4. Cloudy air

a. Virtual potential temperature for cloudy air

The extension of this analysis to define wet virtual adiabats is a little more complex because it involves cloud liquid water as well as vapor. It is still convenient to define from (1a) a virtual potential temperature

$$\begin{aligned} \theta_{vc} &= \theta(1 + 0.61q_s - l) \\ &= T(1 + 0.61q_s - l)(1000/p)^{R/C_p} \end{aligned} \quad (13)$$

where a subscript c has been added to denote that the parcel is cloudy. However unlike (2), θ , q_s and l are not SP functions, so that θ_{vc} , just like θ , is not a conserved parameter for a cloudy parcel, and it changes as a parcel ascends or descends moist adiabatically. Nonetheless, can be defined a wet virtual adiabat (with slight approximation) that joins the SPs of cloudy parcels with the same density or θ_{vc} at the same pressure. It turns out that this wet virtual adiabat bears almost a fixed relationship to the wet adiabat (constant θ_e^*) in (θ, p) coordinates.

b. Wet virtual adiabats

Figure 2 shows schematically these two adiabats, where BA is a wet adiabat (solid line) and BC the wet virtual adiabat (short dashes). Suppose a parcel is lifted from its saturation level, $p^* = 1000$ to $p = 800$ hPa. It's θ follows the wet adiabat BA, and if the condensed liquid water is all precipitated, then, because it is just saturated, the parcel density potential temperature θ_p by definition also follows BA. But if there is no precipitation, then cloud liquid water increases, and θ_p is reduced and follows some line BC. For a cloudy parcel θ_p is also defined as the potential temperature of a saturated parcel (with no liquid) of the same density. If the parcel, on reaching A, has liquid-water content l , then the difference in θ_v with and without liquid l is just

$$\delta\theta_v = \theta_v^*(C) - \theta_v^*(A) = -\theta l = \epsilon\Gamma_w\delta p \quad (14)$$

if the wet adiabat is linearized over a pressure interval δp . This condensed liquid water is itself related to the wet adiabat, and because $\epsilon = C_p T/L \approx 0.12$, the wet virtual adiabat has about 90 percent of the slope of Γ_w . More precisely, by substituting the Clausius-Clayron equation in (6), AC satisfies

$$\delta\theta_v = \delta\theta_p(1 + 0.61\alpha + 0.61q_s), \quad (15)$$

so that the slope of the moist virtual adiabat (linearized over δp)

$$\Gamma_{wv} = (\partial\theta_p/\partial p)_{BC} = \Gamma_w(1 - \beta_2) \quad (16)$$

where

$$\beta_2 = \epsilon/(1 + 0.61\alpha + 0.61q_s).$$

TABLE 3. Temperature ($^{\circ}\text{C}$) on the wet virtual adiabats (constant θ_{ev}^*).

p (mb)	θ_{ev}^* (K)								
	290	300	310	320	330	340	350	360	370
500	-37.29	-30.42	-24.17	-18.61	-13.75	-9.55	-5.92	-2.76	0.02
600	-25.83	-19.36	-13.68	-8.76	-4.54	-0.90	2.26	5.03	7.48
700	-16.45	-10.50	-5.38	-1.00	2.76	6.01	8.85	11.35	13.58
800	-8.69	-3.24	1.41	5.38	8.80	11.77	14.38	16.70	18.77
900	-2.16	2.88	7.14	10.80	13.96	16.72	19.16	21.34	23.29
1000	3.45	8.11	12.09	15.50	18.47	21.08	23.39	25.45	27.03

is allowed to vary along the dotted line AD, there is a saturation point at D on the wet virtual adiabat BC that has the same θ_{vc} and θ as points C and B. Thus, if $\delta\theta_{vc} = 0$ in (21), the slope of the wet virtual adiabat is defined in terms of θ_e^* and q^* as

$$(\partial\theta_e^*/\partial q^*)_{\theta_{vc}} = \eta \quad (22)$$

where

$$\eta = \theta_e^*(1 + \alpha/\epsilon)/(1 + 1.61\alpha + 0.61q_s - l). \quad (23)$$

However, the coefficients involving α , ϵ , q_s , and l are not functions of the conserved variables, so η does not have a unique value at an SP [unlike, say, (6) for the dry virtual adiabats]. The quantity

$$0.61q_s + l \ll 1.61\alpha$$

can be neglected to give

$$\eta = \theta_e^*(1 + \alpha/\epsilon)/(1 + 1.61\alpha). \quad (23')$$

In (23) and (23'), α and ϵ are defined at pressure p not p^* , but if α , ϵ are approximated by their SP values, then (23) and (23') can be integrated over small p^* ranges to define a virtual equivalent potential temperature, which is approximately conserved on a wet virtual adiabat. Because of this approximation [dominated by $\alpha(p) \approx \alpha(p^*)$], it is not useful to integrate (23) from $q^* = 0$. Instead Betts (1983) suggested labeling the wet virtual adiabats with the value of θ_e^* at 1000 mb to give

$$\theta_{ev}^* = \theta_e^* + \int_{p^*}^{1000} \eta \delta q^*. \quad (24)$$

The integral is along the wet virtual adiabat from p^* to 1000 mb. In fact, because virtual equivalent potential temperature θ_{ev}^* can only be defined with approximation, the same accuracy is attainable by taking η outside the integral and defining a more convenient parcel value, where α and ϵ are defined locally.

The quantity

$$\theta_{ev}^* = \theta_e^*[1 + (q_s(\theta_{ev}^*, 1000) - q^*)(1 + \alpha/\epsilon)/(1 + 1.61\alpha)]. \quad (25)$$

This formula fits the wet virtual adiabats for $600 < p^* < 1000$ hPa (and a similar range in p) to <0.4 K in

θ_{ev}^* and <0.2 K in θ_{vc} . Iteration is needed to find the value of q_s at 1000 hPa for $\theta_e^* = \theta_{ev}^*$. This formula is recommended to define the wet virtual adiabats for the lower troposphere. Table 3 tabulates temperature as a function of pressure on the wet virtual adiabats computed from (25).

5. Summary

The convention of plotting a density temperature on a thermodynamic diagram has been introduced; that is, the temperature of a saturated parcel with the same density and virtual temperature as unsaturated or cloudy air parcels at the same pressure. This is similar to the convention of plotting the dewpoint—the temperature of a saturated parcel which has the same mixing ratio as an air parcel. By plotting the density temperatures of air parcels at each pressure, one can compare the buoyancy of cloud and environmental parcels.

Dry and wet virtual adiabats which pass through the parcel density temperatures and saturation points have been defined. These represent the paths traced on a thermodynamic diagram by the density temperatures of unsaturated and cloudy air as these move adiabatically in the atmosphere. They also represent the neutral stability adiabats for comparing the densities of different unsaturated and cloudy air parcels at different levels in the atmosphere. The computation of the dry virtual adiabats is straightforward. An approximate formula is also given for a virtual-equivalent potential temperature, which is conserved on a wet virtual adiabat, and tables were presented for the dry and wet virtual adiabats. Their slopes have been compared with the slope of the wet adiabat, $\Gamma_w = (\partial\theta/\partial p)_{\theta_e^*}$. The dry virtual adiabat has a slope that becomes a larger fraction of Γ_w at warm temperatures, while the wet virtual adiabat has almost exactly nine-tenths of the slope of Γ_w at all temperatures and pressures in the lower troposphere.

On thermodynamic diagrams, and particularly conserved parameter diagrams, overlays of the dry and moist adiabats are useful in assessing stability, given a thermodynamic profile of air parcel SP. The difference between the slope of the dry adiabat and the dry virtual adiabats becomes significant in atmospheres which have large vertical gradients of moisture (such as the

environment of severe storms, Betts 1982). In the tropics the lower troposphere appears to approach neutrality with respect to the wet virtual adiabat.

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