RELATION BETWEEN EQUILBRIUM EVAPORATION AND THE SATURATION PRESSURE BUDGET

ALAN K. BETTS
RR3 Box 3125, Pittsford, VT 05763, USA

(Received 9 February, 1994)

Abstract. The limiting surface Bowen ratios are calculated which maintain mixed-layer saturation pressure and relative humidity, both with and without boundary-layer entrainment. The equations are formally the same as those of Culf (1994) for equilibrium evaporation, but differ numerically because the coefficients are calculated at the mixed-layer saturation temperature (at the lifting condensation level) rather than at the surface temperature. The diurnal cycle over land is used to illustrate the value of these constraints.

1. Introduction

The recent paper (Culf, 1994), on equilibrium evaporation beneath a growing convective boundary layer, is a step forward in understanding surface evaporation models and their links to boundary layer (BL) entrainment. Culf uses the BL model formulation from Betts (1992), so I would like to extend my earlier work, and relate Culf (1994) to the saturation pressure budget for a mixed layer. For air near the surface, saturation pressure p^* corresponds to the familiar lifting condensation level (LCL) pressure, the level at which a parcel that is lifted dry adiabatically (conserving potential temperature, θ and mixing ratio, q) becomes saturated. The fluxes at the earth's surface and through the top of the mixed layer both change the balance of p^* in the mixed layer. Over the oceans, a steady state is often achieved between surface evaporation and the mixing down of dry air (Betts and Ridgway, 1989), giving nearly constant p^* and cloud-base (typically with an LCL height of $\approx$500 m). This balance also involves the radiative cooling of the moist marine layer. Over land, the cooling of the surface at night brings air near the surface close to saturation, and this is followed by the daytime heating cycle in which p^* for the mixed layer decreases so that an afternoon cloud base may be typically 1500 m above the surface. The saturation pressure difference $\mathcal{P} = p^* - p$ is very closely related to relative humidity (RH) (see Figure 1, later), so that the p^* budget can be regarded (if surface pressure is constant) as a measure of RH. The mixed-layer budget equations for p^* give solutions which are related to the "equilibrium evaporation" problem.

Boundary-Layer Meteorology **71:** 235–245, 1994.

2. Mixed-Layer Budgets

2.1. Potential temperature and mixing ratio

Betts (1992) presented the simplified mixed-layer budgets of potential temperature, θ, and mixing ratio, q as (ignoring horizontal advection)

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{F_{s\theta}}{h}\left[1 + \frac{\beta_i}{\beta_s} A_R \frac{(\beta_s - \beta_v)}{(\beta_i - \beta_v)}\right], \tag{1a}$$

$$\rho \lambda \frac{\partial q}{\partial t} = \frac{F_{sq}}{h}\left[1 + A_R \frac{(\beta_s - \beta_v)}{(\beta_i - \beta_v)}\right], \tag{1b}$$

where $\beta_v \simeq -0.07$ is the slope of the dry virtual adiabat (Betts and Bartlo, 1991) on a $(C_p\theta, \lambda q)$ diagram, β_s and β_i are surface (subscript s) and inversion level (subscript i) Bowen ratios, A_R is an entrainment closure parameter, and F_s denotes a surface energy flux in W m^{-2}. (The use of F_θ, F_q rather than C_pH, λE for surface sensible and latent heat fluxes is a difference of notation from Culf (1994).) h, ρ, C_p and λ are respectively the depth and mean density of the mixed layer, the specific heat at constant pressure, and the latent heat of vaporization. The leading terms in Equations (1a) and (1b) are the surface sensible and latent heat fluxes, which warm and moisten the ABL. The second pair of terms, proportional to A_R, are the entrainment fluxes of typically warm dry air at the inversion. Equations (1a), (1b) can be converted using the surface energy budget (surface net radiation, R_n, and ground storage, G), with the small approximation $(\theta/T) \approx 1$

$$R_n - G = F_{s\theta} + F_{sq} = F_{sq}(1 + \beta_s) = F_{s\theta}\frac{(1+\beta_s)}{\beta_s}, \tag{2}$$

to give

$$\rho C_p \frac{\partial \theta}{\partial t} = \frac{(R_n - G)}{(\beta_s + 1)}\left[\beta_s + \beta_i A_R \frac{(\beta_s - \beta_v)}{(\beta_i - \beta_v)}\right], \tag{3a}$$

$$\rho \lambda \frac{\partial q}{\partial t} = \frac{(R_n - G)}{(\beta_s + 1)}\left[1 + A_R \frac{(\beta_s - \beta_v)}{(\beta_i - \beta_v)}\right]. \tag{3b}$$

2.2. Saturation pressure budget

Since saturation pressure p^* is a function of (θ, q), one can expand

$$\delta p^* = \left(\frac{\partial p^*}{\partial \theta}\right)_q \delta\theta + \left(\frac{\partial p^*}{\partial q}\right)_\theta \delta q = \frac{1}{C_p}\left(\frac{\partial p^*}{\partial \theta}\right)_q (C_p\delta\theta - \beta_{p^*}\lambda\delta q), \tag{4}$$

where

$$\beta_{p^*} = (C_p/\lambda)(\partial\theta/\partial q)_{p^*}$$

is the slope of the pressure lines on a $(C_p\theta, \lambda q)$ diagram. Substituting Equation (4) in Equation (3a), Equation (3b) gives the corresponding budget equation for p^*.

$$\frac{\partial p^*}{\partial t} = \frac{1}{C_p}\left(\frac{\partial p^*}{\partial \theta}\right)\frac{(R_n - G)}{(\beta_s + 1)}\left[(\beta_s - \beta_{p^*}) + (\beta_i - \beta_{p^*})A_R\frac{(\beta_s - \beta_v)}{(\beta_i - \beta_v)}\right]. \tag{3c}$$

In Equation (3c), the terms in $(\beta_s - \beta_p)$, $(\beta_i - \beta_p)$ may be thought of as the projection of the surface and inversion fluxes onto the p^* isopleths, using the vector diagram concept discussed in Betts (1992) and Culf (1994). Now

$$\beta_{p^*} = \frac{C_p}{\lambda}\left(\frac{\partial \theta}{\partial q}\right)_{p^*} = \frac{C_p}{\lambda s^*} = \frac{1}{\epsilon^*}, \tag{5}$$

where

$$s^* = \left(\frac{\partial q_s}{\partial \theta}\right)_{p^*} \approx \left(\frac{\partial q_s}{\partial T}\right)_{p^*}, \tag{6}$$

since

$$\left(\frac{\theta}{T}\right) \approx 1 \text{ near } 1000 \text{ mb}.$$

We have added the superscript * to denote a variable calculated at the saturation level p^*, T^*.

Thus the result of Priestley and Taylor (1972), McNaughton (1976), McNaughton and Jarvis (1983) and others is recovered; that if an air mass moves over a region of uniform wetness, the specific humidity deficit at the surface tends towards an equilibrium value, and the surface Bowen ratio $\beta_s \to 1/\epsilon$ (Culf, 1994, Equation (1)) can be written as just the condition $\beta_s \to \beta_p$, where β_p is calculated at (T, p_0). To the extent that we approximate s or $1/\epsilon$ as locally constant, $\beta_p = \beta_{p^*}$. Figure 1 makes clear both the relationsihps between different variables, and the nature of the constant ϵ approximation. For small $\mathcal{P}$ (large RH), the variations of ϵ and s are small, and the pressure lines are nearly parallel. Constant potential saturation humidity deficit $(q^*(\theta) - q)$ then corresponds to constant p^*, as well as constant dewpoint depression, and constant relative humdity (RH). However, because ϵ is a strong function of T, it is significant whether β_p is calculated at T or T^*, and if $p^* < p_0$, $\beta_{p^*} \neq \beta_p$. This is discussed in the next section.

There is a similar equation for the equivalent potential temperature (θ_E) budget with β_w replacing β_p. The slope of the θ_E lines on Figure 1 is

$$\beta_w = \frac{C_p}{\lambda}\left(\frac{\partial \theta}{\partial q}\right)_{\theta_E} = -\frac{\theta}{T} \simeq -1\,. \tag{7}$$

Indeed the coefficient $(\beta_s + 1)$ in Equation (3) may also be regarded as $(\beta_s - \beta_w)$ to this level of approximation.

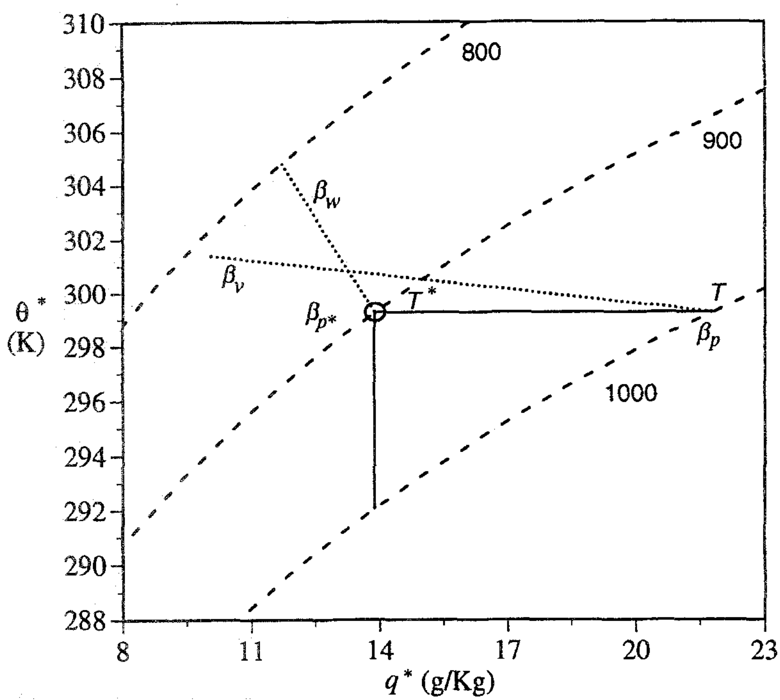

Fig. 1. (θ, q) diagram showing slope of saturation pressure lines at different pressures and mixing ratios.

2.3. Difference between Equilibrium Evaporation Models Based on β_p at T and T^*

There is an important formal difference between the assumption of constant p^* (which also means very closely constant RH), and the assumption of constant $(q^*(\theta) - q)$ made by Culf (1994), and the literature which preceded it, including Priestley and Taylor (1972), Monteith (1981), De Bruin (1983), McNaughton and Jarvis (1983) and McNaughton and Spriggs (1986) and others. The coefficient β_p^* (and corresponding ϵ^* in Equation (3c) is calculated at the *saturation point* (see Betts, 1982), which in this case is (T^*, p^*), the temperature and pressure at the LCL. The analysis of Culf (1994) implies that the calculation of s, ϵ corresponds to $q_s(\theta)$, which is at $T(\theta, p_0)$, the temperature on the dry adiabat at the surface pressure. If variations of ϵ are neglected, this difference is also neglected. However, for afternoon mixed-layer depths of $\approx$1500 m, $T - T^* \approx 15$ K, and the difference in $\beta_{p^*} - \beta_p$ is significant.

The p^* analysis, presented in this note, explicitly calculates ϵ^* and β_{p^*} at the mixed-layer saturation point. Since Culf (1994) simply shows results for a range of temperature, our results can be compared numerically by regarding his temperatures as saturation level temperature, T^*. It could be argued that my use of p^*, T^* is inconsistent with the long history of surface models, which naturally use surface temperature in the calculation of ϵ, because surface evaporation is related to it. However, from a theoretical viewpoint, the introduction of entrainment and the use of mixed-layer budget analysis using conserved variables, implies the use of saturation point variables (Betts, 1982). I shall first explore the consequences of assuming equilibrium p^* (or RH) in the mixed-layer budget. This gives solutions that are mathematically identical to Culf (1994), although they differ numerically because $T^* < T$. I shall then ask whether either approach gives a clearer understanding of the diurnal cycle over land, which is very far from equilibrium.

3. Derivation of Culf (1994) Equation (23) from the p^* Budget

Culf (1994) showed that the entrainment terms in the mixed-layer budget (e.g., Equations (1) or (3) above) gave a smaller surface Bowen ratio corresponding to "equilibrium evaporation". This result can be simply derived by setting $\partial p^*/\partial t = 0$ in Equation (3c) to give

$$(\beta_s - \beta_{p^*}) = (\beta_i - \beta_{p^*})A_R(\beta_s - \beta_v)/(\beta_i - \beta_v), \tag{8}$$

$$\therefore\ \beta_s = (\beta_{p^*} + \beta_v A_R \xi)/(1 + A_R \xi), \tag{9}$$

where

$$\xi = (\beta_i - \beta_{p^*})/(\beta_i - \beta_v). \tag{10}$$

Our Equation (9) is just Culf's Equation (23) with $\beta_{p^*} = 1/\epsilon^*$, calculated at the saturation level T^*, p^*. Indeed Equation (9) is completely transparent, if it is derived graphically from Culf (1994), Figure 2. The two constraints of the mixed-layer model can be written as

$$A_R\ \text{closure}: \quad -F_{sq}(\beta_s - \beta_v)A_R = F_{iq}(\beta_i - \beta_v),$$

$$p^*\ \text{balance}: \quad F_{sq}(\beta_s - \beta_{p^*}) = F_{iq}(\beta_i - \beta_{p^*}).$$

Division gives Equation (9) again. In this form Equation (9) involves the graphical projection of the fluxes onto the θ_v and p^* lines of Culf's Figure 2 (and some similar triangle geometry).

One further substitution simplifies Equation (9). Let $M = A_R\xi$ be a measure of the impact of the BL-top entrainment on the p^* budget of the mixed layer. Equation (9) then becomes

$$\beta_s = (\beta_{p^*} + M\beta_v)/(1 + M) = \beta_{p^*} - \frac{M}{1+M}(\beta_{p^*} - \beta_v). \tag{11}$$

For the FIFE data, which we shall present later, $\beta_i \approx -0.3$, $\beta_{p^*} \approx 0.5$, $\beta_v = -0.07$, so that if $A_R \approx 0.4$, $M \approx 1.4$. We see that the effect of entrainment (M) is

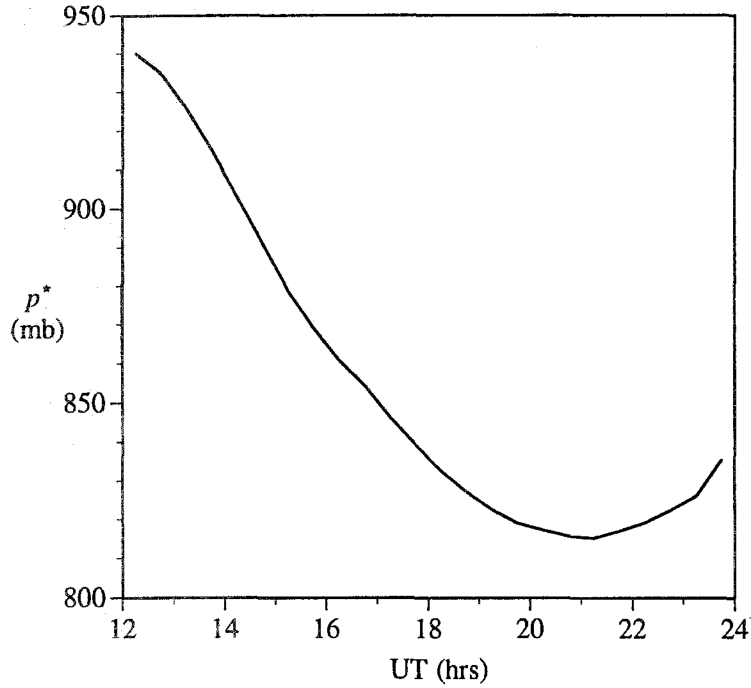

Fig. 2. Variation of saturation pressure, p^*, during the daytime diurnal cycle, calculated from a FIFE-1987 average.

to reduce the surface Bowen ratio needed to maintain p^* to below β_p. Substituting Equations (11) in (2) gives an equilibrium evaporative fraction (Culf's Equation (24)) for this β_s

$$EF^* = \frac{F_{sq}}{R_n - G} = \frac{(1+M)}{1+\beta_{p^*}+M(1+\beta_v)}. \tag{12}$$

A corresponding equilibrium "Priestley–Taylor" parameter which includes BL-top entrainment can therefore be written

$$\alpha_M^* = \frac{1+M}{1+M\left(\frac{1+\beta_v}{1+\beta_{p^*}}\right)}. \tag{13}$$

Substituting $\beta_{p^*} \approx 0.5$ and $\beta_v = -0.07$, simplifies Equation (13) to

$$\alpha_M^* = (1+M)/(1+0.62M). \tag{13$'$}$$

We see that $\alpha_M^* > 1$, and is directly related to the effect of entrainment on the p^* budget.

However, although our equations are formally the same as those of Culf, the numerical and conceptual difference is significant, because of the temperature dependence of β_p through $s(T)$. A surface evaporation corresponding to

$\beta_s = \beta_{p^*}$ maintains p^* and RH constant in the absence of entrainment. A lower surface Bowen ratio, given by Equations (9) or (11) (and higher evaporative fraction given by Equation (12)), maintains constant p^* and RH in the presence of entrainment of air from above the mixed layer. The analysis of Culf (1994) and the antecedent papers (loc. cit.) discuss the surface equilibrium evaporation, which maintains $(q_s(\theta)) - q)$, rather than p^* and RH. In the absence of entrainment, this corresponds to β_p (calculated at (T, p_0); while with entrainment, it is still given by Equation (9), but with β_p replacing β_{p^*}. However, these solutions for β_p, β_s, EF, α, are different numerically because $T - T^*$ can become as large as 15 K in the afternoon over land, and they correspond to different reference processes. Is either of them more useful? We have shown that the p^* budget analysis follows directly from the mixed-layer constraints. Its main convenience is meteorological, as it is closely coupled to the meteorological forcing. The traditional equilibrium evaporation analysis makes several assumptions. It neglects the differences and changes between the leaf surface potential temperature, that of the air at say 2 m, and that of the mixed layer (i.e., it neglects the entire superadiabatic layer). However, the differences across the superadiabatic layer, although significant, are themselves smaller than $T - T^*$. The equilibrium evaporation derivations usually assume (e.g., Culf, 1994), constant surface resistance to evaporation, although vegetative resistance is known to vary with both RH and T (e.g., Collatz *et al.*, 1991). The diurnal cycle is so large over land, that it is unclear whether the assumptions of traditional equilibrium evaporation models are satisfied. The mixed-layer model analysis is not based on assumptions at the leaf level. It also has a further advantage; the diurnal variation of T^* is small in comparison with the variation of T, so that β_{p^*} and EF^* give nearly constant reference processes, which can be compared with observed β_s and evaporative fraction.

4. Illustration of p^* Model

Whether any "equilibrium" solutions are widely applicable over land needs discussion. Surface evaporation over land is rarely sufficient to keep saturation pressure p^* constant (i.e., β_s as small as given by Equations (9)). Typically p^* (familiar as the LCL) rises rapidly during the morning diurnal cycle (see Figure 2 below), and only reaches quasiequilibrium in the early afternooon, as the surface heat flux falls and the superadiabatic layer weakens. In addition over moist terrain, the growing BL-top reaches p^* (the LCL) and this modifies the subsequent rise of the BL-top, because p^* and h become coupled as small cumulus begin to form, modulating the cloud-base fluxes (and β_i). There is therefore some inherent incompatibility between a BL growing by extrainment, and p^* remaining constant: the LCL will soon be reached, and then p^* and h grow together. However, the limit of $\partial p^*/\partial t = 0$, and the associated β_s and EF^* given by Equations (11), (12) are of theoretical interest, as they suggests the atmospheric factors which limit evaporation. Figure 2 shows the daytime change of p^* based on a FIFE-1987 (FIFE was the First ISLSCP (International Satellite Land Surface Climatology Project) Field Experiment) average, calculated from near-surface T, q measurements at 2 m elevation. The mean surface pressure is 968 mb, and local noon is 1820 UT. There are 64 days in the average. These days (between June and September) were selected on the basis of having a mean surface Bowen ratio near local noon of less than 0.5; this means they correspond generally to days of unstressed evaporation, when soil moisture is high. In the afternoon, p^* appears to reach a quasi-steady value (Figure 2) corresponding to an LCL near 820 mb. Are the solutions given by Equation (9) relevant for the time period 20–22 UT? Probably not, because p^* is calculated from 2 m elevation measurements, not mixed-layer means. The p^* variation shown is fairly representative of the mixed layer as long as T is increasing, but not once the surface starts to cool. This is exactly the time when p^* (2-m) becomes constant, so it is unclear that Equation (9) applies.

However, we have corresponding measurements of the surface energy budget (derived from the time series of the two Bowen ratio stations discussed in Smith *et al.* 1992), so we can calculate the observed evaporative fraction

$$EF = \frac{F_{sq}}{R_n - G}, \tag{14}$$

and compare it with the two theoretical limits. The lower limit which ignores entrainment is

$$EF^*(M = 0) = \frac{1}{(1 + \beta_{p^*})} \approx 0.67, \tag{15}$$

if we substitute $\beta_{p^*} = 0.5$. The upper limit which includes entrainment is given by Equation (12). To estimate M, we set $A_R = 0.4$, $\beta_i = -0.3$ (Betts, 1992; Betts and Ball, 1994), which gives $M = 1.4$, and $\alpha_M^* \approx 1.28$, so that we can estimate

$$EF^* = \alpha_M^* EF^*(M = 0) \approx 1.28 * 0.67 = 0.86. \tag{16}$$

The solid line in Figure 3 is the observed EF given by Equation (14), the dashed lines are the two theoretical limits given by Equations (12) and (15), and the dotted line is the 2-m RH which falls to a minimum near 2100 UT. Note that the dashed lines are nearly constant (since T^* varies little during the day), and they bracket EF, the observed evaporative fraction, during much of the day. This is understandable. Because $EF > EF^*$ $(M = 0)$, without entrainment the mixed layer would move towards saturation. Entrainment shifts the important reference to EF^*; and since $EF < EF^*$, the surface evaporation is insufficient to prevent p^* from rising until very late in the day, around 2100 UT (although by then the surface has started to cool and $\partial p^*/\partial t$ calculated from 2-m data is unrepresentative of the mixed-layer change).

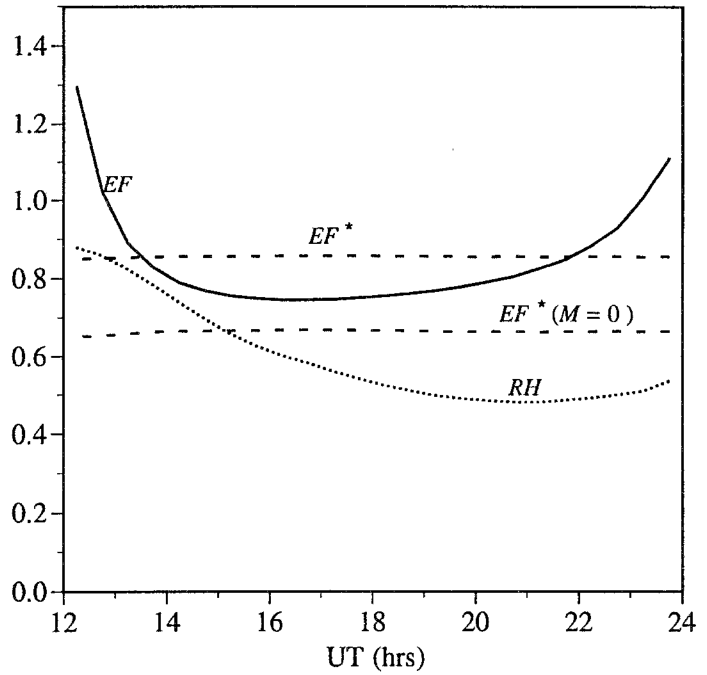

Fig. 3. Daytime evaporative fraction for FIFE-1987 average compared with two reference evaporative fractions that maintain p^*, both with and without entrainment.

5. Calculation of Priestley–Taylor Parameters with Different Reference Temperatures

The difference between calculating β_p and a corresponding Priestley–Taylor parameter at T or T^* is large (see Figure 4). The solid lines come from Figure 3: the ratio is

$$\alpha_M^* = EF^*/EF^*(M = 0),$$

and a corresponding value for the data

$$\alpha_D^* = EF/EF^*(M = 0).$$

These can be regarded as simply a replot of the corresponding evaporative fractions from Figure 3. The dotted lines however are corresponding curves for α_M, α_D computed consistently using β_p, which is less than β_{p^*} because $T > T^*$. Note that $\alpha_M < \alpha_M^*$, and it has a greater diurnal variation as T increases. Note that unlike α_D^*, α_D actually becomes < 1. Numerically this just means $\beta_p < \beta_s < \beta_p^*$. Conceptually it shows the marked difference between the two reference processes. $\beta_s < \beta_p^*$ means that the surface fluxes acting alone would move the relatively dry mixed layer (with saturation level at p^*) towards saturation. However, the surface is sufficiently warm that $\beta_p < \beta_s$, and the surface

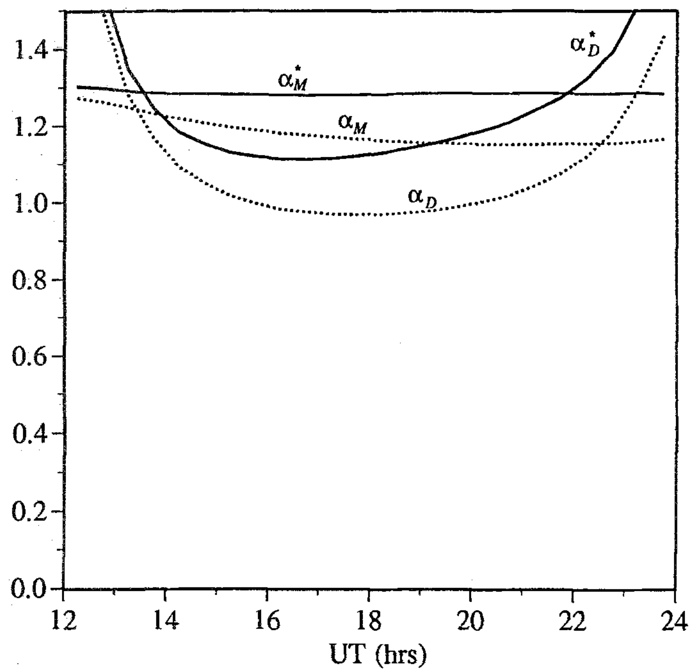

Fig. 4. Daytime Priestley–Taylor parameters derived from Figure 3, with β_p calculated at air temperature (dotted) and LCL temperature (solid).

fluxes acting alone would not stop $(q_s(\theta) - q)$ from rising. Given that all the variables that are changing (net radiation, surface and skin temperature, vegetative and aerodynamic resistance among others), it is clear that "equilibrium evaporation" is not easy to define.

6. Conclusions

My conclusion is that although the diurnal cycle over land is very far from a steady state equilibrium, the saturation pressure budget analysis is probably more useful for interpretation. There are two reference surface evaporative fractions, which would give p^* equilibrium with and without BL-top entrainment. Typically the surface EF is between these during the peak of the daytime heating cycle over moist land surfaces. Thus, although the downward mixing of dry air increases surface evaporation, this is not sufficient to prevent p^* or RH from falling. That is, $EF^*(M = 0) < EF < EF^*$. Since p^* needs to fall during the daytime over land to compensate for its rise at night by radiative cooling, this result is not unexpected. The afternoon evaporation over wet soils does come close to an equilibrium evaporation model. I have interpreted this model however as the condition of constant saturation pressure or relative humidity. Only to the extent that $\beta_p = (C_p/\lambda)(\partial\theta/\partial q)_p$ is assumed constant (a poor assumption), is this the same as the constant saturation deficit model. Clearly this needs further study, but it is likely that the limits derived from the saturation pressure model will be more useful, because of their close connection to the mixed-layer budgets.

Acknowledgments

This work has been supported by NASA-GSFC under contracts NAS5-31738 and NAS5-32334 and NSF under Grant ATM90-01960.

References

Betts, A. K.: 1982, 'Saturation Point Analysis of Moist Convective Overturning', *J. Atmos. Sci.* **39**, 1484–1505.

Betts, A. K.: 1992, 'FIFE Atmospheric Boundary Layer Budget Methods', *J. Geophys. Res.* **97**(D17), 18,523–18,531.

Betts, A. K. and Ball, J. H.: 1994, 'Budget Analysis of FIFE-1987 Sonde Data', *J. G. R.* **99**, 3655–3666.

Betts, A. K. and Bartlo, J.: 1991, 'The Density Temperature and the Dry and Wet Virtual Adiabats', *Mon. Wea. Rev.* **119**, 169–175.

Betts, A. K. and Ridgway, W. L.: 1989, 'Climatic Equilbrium of the Atmospheric Convective Boundary Layer over a Tropical Ocean', *J. Atmos. Sci.* **46**, 2621–2641.

Collatz, G. J., Ball, J. T., Grivet, C., and Berry, J. A.: 1991, 'Physiological and Environmental Regulation of Stomatal Conductance, Photosynthesis and Transpiration: A Model that Includes a Laminar Boundary Layer', *Agric. For. Meteorol.* **54**, 107–136.

De Bruin, H. A. R.: 1983, 'A Model for the Priestley–Taylor Parameter α', *J. Clim. Appl. Meteor.* **22**, 572–578.

Culf, A. D.: 1994, 'Equilibrium Evaporation Beneath a Growing Convective Boundary Layer', *Boundary-Layer Meteorol.* **70**, 37–49.

McNaughton, K. G.: 1976, 'Evaporation and Advection I: Evaporation from Extensive Homogeneous Surfaces', *Quart. J. Roy. Meteorol. Soc.* **102**, 181–191.

McNaughton, K. G. and Jarvis, P. G.: 1983, 'Predicting Effects of Vegetation Changes on Transpiration and Evaporation', in T. T. Kozlowski (ed.), *Water Deficits and Plant Growth*, Vol. VII, pp. 1–47. Academic Press, New York.

McNaughton, K. G. and Spriggs, T. W.: 1986, 'A Mixed Layer Model for Regional Evaporation', *Boundary-Layer Meteorol.* **34**, 243–262.

Monteith, J. L.: 1981, 'Evaporation and Surface Temperature', *Quart. J. R. Meteor. Soc.* **107**, 1–27.

Priestley, C. H. B. and Taylor, R. J.: 1972, 'On the Assessment of Surface Heat Flux and Evaporation', *Mon. Weath. Rev.* **106**, 81–92.

Smith, E. A., Crosson, W. L., and Tanner, B. D.: 1992, 'Estimation of Surface Heat and Moisture Fluxes over a Prairie Grassland. 1. *In situ* Energy Budget Measurements Incorporating a Cooled Mirror Dewpoint Hygrometer', *J. Geophys. Res.* **97**, 18,557–18,582.