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The energy formula in a moving reference frame

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SUMMARY

The energy formula quoted by Green, Ludlam and McIlveen (1966) requires an additional term if it refers to a frame moving over the Earth's surface. The correction is derived for steady motion of the frame and tested observationally. The error incurred by ignoring the motion of a weather system is discussed.

1. INTRODUCTION

The analysis of airflow in a large-scale weather system is greatly simplified if the system can be assumed to be stationary and in a steady state when observed from some reference frame moving steadily over the Earth's surface (Green *et al.*, loc cit.). When this is so, streamlines of the flow relative to the moving system are also trajectories. Green *et al.* used a relation between kinetic and available potential energy to find the speed of air in a jet-stream, given the path along which the air ascended from near sea-level in lower latitudes. In the cases studied there and by Ludlam (e.g. 1967), the systems were virtually stationary or the latitudinal separation of the ends of the trajectories were small, and the speeds within the jet-streams were rather accurately predicted. However, it was mistakenly stated by Green *et al.* that the energy formula as written there was still applicable if the system moved steadily and was in a steady state – a statement doubted by W. T. Roach in a private communication.

By considering a reference frame in steady horizontal motion over the Earth's surface (discussed briefly by Eady 1949), the present note derives and discusses the energy formula for observations from the moving frame, and investigates the effects on kinetic energy computations of neglecting the motion of a steady-state system, finding them negligible in some but not all realistic situations.

2. THE ENERGY FORMULA IN A MOVING FRAME

The Coriolis acceleration in the momentum equation is

$$2\,\boldsymbol{\omega} \wedge \mathbf{V}$$

where $\boldsymbol{\omega}$ is the Earth's angular velocity and $\mathbf{V}$ is the wind velocity relative to the Earth.

If $\mathbf{V}$ is replaced by $\mathbf{V}_r + \mathbf{U}$, where $\mathbf{V}_r$ is the wind velocity relative to a frame moving with constant uniform velocity $\mathbf{U}$, the expression

$$2\,\mathbf{V}_r \cdot (\boldsymbol{\omega} \wedge \mathbf{U}) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

will remain after making a scalar product of the momentum equation with $\mathbf{V}_r$, and hence appear in the expression for the rate of change of specific energy observed from the moving frame.

The total specific energy of an air parcel can be found in the usual manner to satisfy

$$\frac{D}{Dt}\left[\tfrac{1}{2}V_r^2 + gZ + c_p T - Lx\right] + 2\,\mathbf{V}_r \cdot (\boldsymbol{\omega} \wedge \mathbf{U}) = \frac{1}{\rho}\left(\frac{\partial p}{\partial t}\right)_r \quad . \quad . \quad (2)$$

where subscript r denotes a measurement relative to the moving frame, Z is the geopotential height and T the temperature of the parcel, x is the mixing ratio of condensed water (printed with the wrong sign in Eqs. (1) and (2) of Green *et al.*), and g, c_p and L have their conventional meanings.

It is assumed that air behaves as a perfect gas and that the transport of heat, water and momentum to the parcel by radiation and turbulent diffusion can be ignored. When the system is in a steady state relative to the moving frame, the term in $(\partial p/\partial t)_r$ on the right-hand side of Eq. (2) vanishes and the last term on the left-hand side represents the modification needed to allow for the translation $\mathbf{U}$.

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The above treatment is valid for any two Cartesian frames in unaccelerated relative motion but, to be useful, one frame must move parallel to the Earth's curved surface and so accelerate relative to the other fixed in the Earth. The effect of this acceleration is included in the following more complete derivation.

The usual treatment of rotating frames of reference can be extended to apply to the case of a frame rotating with steady angular velocity $\boldsymbol{\omega}_f$ relative to the Earth; the acceleration relative to an inertial frame can then be shown to be

$$\underset{\text{(a)}}{\frac{D}{Dt_r}(\mathbf{V}_r)} + \underset{\text{(b)}}{2(\boldsymbol{\omega} + \boldsymbol{\omega}_f) \wedge \mathbf{V}_r} + \underset{\text{(c)}}{2\boldsymbol{\omega} \wedge (\boldsymbol{\omega}_f \wedge \mathbf{R})} + \underset{\text{(d)}}{\boldsymbol{\omega}_f \wedge (\boldsymbol{\omega}_f \wedge R)} + \underset{\text{(e)}}{\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R})} \quad . \quad (3)$$

where **R** is the radius vector from the Earth's centre. Of these terms,

(b) vanishes on making a scalar product with $\mathbf{V}_r$ in the derivation of the energy formula;

(d) has a magnitude about 1/50 of that of (c) when $\boldsymbol{\omega}_f$ is appropriate to the motion of large weather systems; its effect is comparable with that of the curvature of the conventional co-ordinates (see, for example, Hess 1959, p. 166) and likewise may be neglected;

(e) is the familiar centripetal acceleration which becomes part of the apparent gravitational acceleration observed at a fixed point on the Earth.

Hence, when the reference frame moves at speeds typical of large weather systems, the only term contributing significantly to the expression for the rate of change of specific energy, and containing the motion of the reference frame explicitly, is (c) of the expression (3). Noting that $\boldsymbol{\omega}_f \wedge \mathbf{R}$ is **U**, scalar multiplication by $\mathbf{V}_r$ yields just that term derived in the simpler treatment above, though in the more complete derivation it is $\boldsymbol{\omega}_f$, and not **U**, which is constant.

Clearly we can define an apparent pressure gradient balancing the Coriolis acceleration associated with the motion of the frame. That is

$$f\mathbf{k} \wedge \mathbf{U} = -\nabla(gZ_f) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where k is the unit vector perpendicular to the Earth's surface, and f is the conventional Coriolis parameter (a technique of synoptic analysis of relative flow based on this relation has been used for some time in this Department by R. S. Harwood). Since Z_f is a function of position only, the scalar product of Eq. (4) with $\mathbf{V}_r$ may be inserted in Eq. (2), the left-hand side of which may be integrated with respect to time to give a simple expression for the total specific energy of an air measured from the moving frame (in which the motion system is assumed to be stationary and steady):

$$\tfrac{1}{2} V_r^2 + g(Z - Z_f) + c_p T - Lx = \text{constant} \quad . \quad . \quad (5)$$

This is the correct form of Eq. (2) of Green *et al.*

3. The correction term for zonal frame motion; comparison with observations

A reference frame moving zonally eastwards is suitable for the analysis of many mid-latitude weather systems. Neglecting the trivial influence of vertical motion, we find

$$gZ_f = \tfrac{1}{2}\,\omega\,\omega_f R^2 \cos 2\phi + \text{constant} \quad . \quad . \quad . \quad (6)$$

where ϕ is latitude.

From Eqs. (5) and (6) it is clear that, if the term in Z_f is neglected when an air parcel is moving northward, the increase in specific kinetic energy computed from the remaining terms of Eq. (5) will be too large. For example, if air moves from 40°N to 50°N in a system moving east at 10 m s^{-1} (at 45°N), gZ_f decreases by $11{\cdot}5 \times 10^6$ ergs g^{-1} along the trajectory, which is the energy equivalent of unit mass decelerating to rest from 47 m s^{-1}.

McIlveen (1966) compared rather small changes of specific kinetic energy along trajectories confined to the upper troposphere, estimated from observations of winds, with changes computed using Eq. (2) of Green *et al.* Since in this case a trough was approaching western Europe at

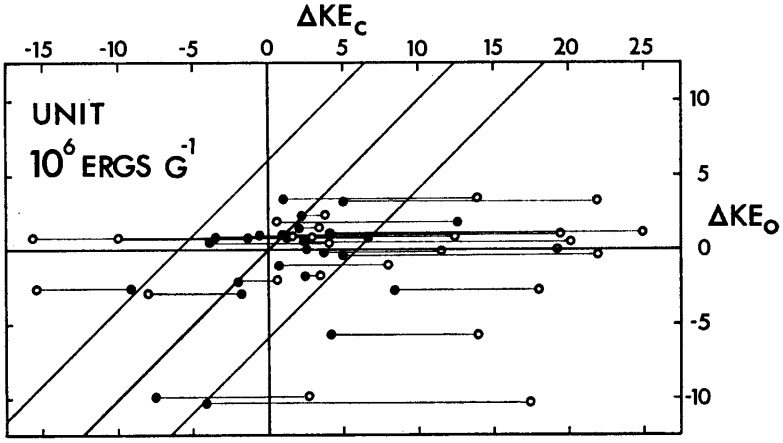

Figure 1. Observed changes of specific kinetic energy of relative motion (KE_0) compared with those computed. Points obtained by including the term in gZ_f in the computation of KE_c (using Eqs. (5) and (6)) are indicated by full circles; they are joined by horizontal lines to the corresponding points obtained by ignoring the term in gZ_f (open circles). The envelope indicates the probable maximum effects of errors of observation and analysis.

12 m s^{-1}, the neglected term in Z_f was important when trajectories spanned more than a few degrees of latitude. As is shown in Fig. 1, 17 out of the 26 changes of specific kinetic energy computed using the incorrect expression differed from those observed by more than 6×10^6 ergs g^{-1} (the maximum discrepancy attributable to inaccuracies of observation and analysis), whereas, using the correct expression (Eqs. (5) and (6)), only three discrepancies significantly exceeded this limit. All large discrepancies were markedly reduced by the incorporation of the term in Z_f and in only one case was a small discrepancy significantly increased.

4. Criteria for ignoring the motion of a system

Putting $\mathbf{V}_r = \mathbf{V} - \mathbf{U}$ in Eq. (5) and intergrating between end points 1 and 2 of a trajectory, three terms of the resulting expression for the change in $\frac{1}{2} V^2$ contain the velocity of translation $\mathbf{U}$. They are

$$[\mathbf{U} \cdot \mathbf{V} - \tfrac{1}{2} U^2 + gZ_f]_1^2 \qquad (7)$$

where square brackets denote the difference in the enclosed terms between the end points 1 and 2 of the trajectory. This expression is equivalent to the term in the local rate of change of pressure with time which appears in the usual derivation of the energy formula for a frame fixed in the earth.

To see the effect on energy computations of ignoring the motion of a weather system, consider again air flowing with a northward component in a steady-state system moving zonally eastward with constant angular velocity. The terms of expression (7) tend to cancel, and do so completely when zonal angular momentum is conserved. The latter conclusion follows when the condition for zero torque of the pressure gradient about the Earth's axis of rotation is inserted in

$$\frac{1}{\rho}\frac{\partial p}{\partial t} = -\frac{1}{\rho}\mathbf{U} \cdot \nabla p$$

(a consequence of the system being steady in the moving frame).

In general the fractional error in V_2 (the computed wind speed at the end point (2) of a trajectory), incurred by ignoring the motion of the frame, may be obtained from expression (7), and, if small, is given by

$$\epsilon = -[Uu - \tfrac{1}{2} U^2 + gZ_f]_1^2 / V_2^2$$

where u is the westerly component of the wind velocity.

In realistic conditions ω_f is much smaller than ω so that the second term in brackets can be ignored in comparison with the third. Neglecting the variation of f with latitude, Eq. (4) gives

$$gZ_f = - Ufy + \text{constant}$$

where y is the distance along a meridian from an arbitrary zero; since the change in u along a trajectory is usually much greater than that in U, we obtain

$$\epsilon = - U\,[u - fy]_1^2/V_2^2. \qquad (8)$$

In the extreme case of conservation of zonal angular momentum, the terms in square brackets cancel and the correction ϵ vanishes, as has been deduced alternatively above. In most realistic conditions, however, the second term in brackets is somewhat larger than the first giving

$$\epsilon \approx \frac{U}{V_2}\left\{\frac{f(y_2 - y_1)}{V_2}\right\}. \qquad (9)$$

In a similar manner it can be shown that the error in the relative wind speed V_{r2} computed using Eq. (5), but while ignoring the term in Z_f, is given by Eq. (9) also.

Thus the movement of the system can be ignored if U is sufficiently smaller than V_2 to outweigh the term involving the meridional displacement (in curly brackets in Eq. (9), and normally at least of order unity). Such conditions are often met on the equatorial flanks of westerly jet-streams where $f(y_2 - y_1) \approx 2\,V_2$; other cases would have to be considered individually.

Green *et al.* computed specific kinetic energies in jet-streams associated with almost stationary weather systems. However, because of the inevitable uncertainties of analysis, it is possible that in those cases the systems were actually moving with speeds of as much as 2 m s^{-1}: Eq. (9) shows that, even assuming such motion, negligible errors are produced in kinetic energy computations by taking the system to be stationary, since speeds in the jet-streams were of order 50 m s^{-1} and the term in curly brackets was of order 2. By contrast, the case considered by McIlveen (loc cit. and Fig. 1) involved a system moving at 12 m s^{-1} and wind speeds of order 30 m s^{-1}. Since meridional displacements were similar to those considered by Green *et al.*, the resulting values of ϵ were large, as is apparent in Fig. 1.

5. Conclusion

The energy formula in a frame rotating steadily relative to the Earth contains a term, explicitly involving this rotation, which reduces to a simple form (Eq. (6)) when the axes of rotation of the frame and the Earth are parallel. In this form the magnitude of the correction has been verified using the analysis of a rapidly moving trough in middle latitudes. The energy formula may be used ignoring the effects of motion of a steady-state weather system (the local rate of change of pressure in a reference frame fixed in the earth, or the additional Coriolis term if the frame is fixed in the moving system) when the error defined by Eq. (9) is small, and in particular along trajectories in the equatorial flanks of westerly jet-streams in slow-moving systems.

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