Empirical Formula for Saturation Pseudoadiabats and Saturation Equivalent Potential Temperature

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Exact water saturation pseudoadiabats can be constructed by the integration of the equation (Saunders, 1957)

$$(C_p+r_sC)\frac{dT}{T}-\frac{Rdp_a}{p_a}+d\left(\frac{L_vr_s}{T}\right)=0, \tag{1}$$

where C_p, R, p_a are the specific heat, gas constant and pressure of dry air; r_s, L_v the saturation mixing ratio and latent heat of vaporization of water; C the specific heat of liquid water; and T the mixture temperature. Alternatively, one may integrate theequivalent equation (e.g., Betts, 1973)

$$C_{pm}\frac{dT}{T}-R_m\frac{dp}{p}+L_v\frac{dq_s}{T}=0, \tag{2}$$

where C_{pm}, R_m, p are the specific heat, gas constant and total pressure of the air-vapor mixture; and q_s is the saturation specific humidity.

The approximate formula for saturation equivalent potential temperature,

$$\theta_{ES}\approx\theta\exp\left(\frac{L_vr_s}{C_pT}\right), \tag{3}$$

can be derived from (1) or (2) (Rossby, 1938; Hess, 1959; Holton, 1972). The same formula (3), with mixing ratio r replacing r_s, has been used to define the equivalent potential temperature of an unsaturated air parcel. The value of (3) is that it is a function only of parcel pressure and temperature. However, its derivation from (2) clearly involves (among other approximations) the neglect of the variation of T in the third term, which decreases as q_s decreases along the pseudoadiabat. Rather than use a value of T^{-1} averaged along the pseudoadiabat with respect to q_s (or approximately r_s), one can match the exact pseudoadiabats to high accuracy by choosing a value of A in the empirical formula

$$\theta_{ES}=\theta\exp\left(\frac{Ar_s}{T}\right). \tag{4}$$

This generates a more accurate formula than (3) for θ_{ES}, but one which is still only a function of the state variables (p,T) of a parcel.

The *Smithsonian Meteorological Tables* (1971) tabulate the water saturation pseudoadiabats for 2C increments of saturation wet bulb potential temperature θ_s, and give corresponding values of θ_{ES} for the same pseudoadiabats. Eq. (4) generates these pseudoadiabats to an accuracy of ±0.2K in θ_{ES} at all pressures, and

TABLE 1.

θ_s (°C)	θ_{ES} (°K)	θ_s (°C)	θ_{ES} (°K)
−20	255.2	10	304.2
−18	257.6	12	309.4
−16	260.0	14	315.0
−14	262.5	16	321.1
−12	265.1	18	327.8
−10	267.8	20	335.2
−8	270.6	22	343.3
−6	273.6	24	352.3
−4	276.7	26	362.3
−2	279.9	28	373.4
0	283.3	30	385.8
2	287.0	32	399.8
4	290.8	34	415.6
6	295.0	36	433.6
8	299.4	38	454.0
		40	477.5

for values of θ_s from -20 to 40C with

$$A = 2.625 + 0.0014\theta_s, \quad (5)$$

for saturation mixing ratios r_s in gm kg^{-1}.

A separate problem is the variability of values used for the specific heat of dry air. The Smithsonian pseudoadiabat tabulations are apparently based on $R=2.871 \times 10^6$ ergs gm^{-1} (°K)$^{-1}$, $C_p=0.238$ cal gm^{-1} (°K)$^{-1}$, giving $\epsilon = R/C_p = 0.288$. However, at low pressure, where θ_{ES} becomes θ, the tabulated values require $\epsilon = 0.2875$. Using the currently recommended value for C_p of 0.240 cal gm^{-1} (°K)$^{-1}$ (*loc. cit.*, p. 289) and the corresponding $\epsilon = 0.286$ gives slightly different pseudoadiabats. A corresponding formula for A in (4) is

$$A = 2.61 + 0.0014\theta_s. \quad (6)$$

The variation of A with θ_s (or θ_{ES}) is rather small, and can be neglected for some purposes. Given an exact tabulation, the ice saturation pseudoadiabats could be approximated using (4), and a different value for A.

Table 1 shows the conversion between θ_s and θ_{ES} based on Eqs. (4) and (6).

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